Chapter 2: Electric Energy and Capacitance

One goal of physics is to identify basic forces in our world, such as the electric force as studied in the previous lectures. Experimentally, we discovered that the electric force is conservative and thus has associated electric potential energy. Therefore, we can apply the principle of the conservation of mechanical energy for the case of the electric force. This extremely powerful principle allows us to solve problems for which calculations based on the force alone would be very difficult.
Chapter 2: Electric Energy and Capacitance

2.1. Electric Potential and Electric Potential Difference
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2.1. Electric Potential and Electric Potential Difference:

2.1.1. Electric Potential Energy:

**Key issue:** Newton’s law (for the gravitational force) and Coulomb’s law (for the electrostatic force) are mathematically identical. Therefore, the general features for the gravitational force is applicable for the electrostatic force, e.g. the properties of a conservative force

- We can assign an **electric potential energy** $U$:

\[
\Delta U = U_f - U_i = -W
\]

$W$: work done by the electrostatic force on the particles

\[
W = -W_{\text{applied}}
\]

- The reference configuration ($U = 0$) of a system of charged particles: all particles are infinitely separated from each other

Thus, the potential energy of the system: $W_\infty$: work done by the electrostatic force during the move in from infinity

\[
U = -W_\infty
\]
Checkpoint 1: A proton moves from point i to f in a uniform electric field directed as shown: (a) does E do the positive or negative work on the proton? (b) does the electric potential energy of the proton increase or decrease?

(a) Work done by the electric field E:

\[
W = \vec{F} \cdot \vec{d} = q\vec{E}\vec{d} = qEd \cos \theta = -qEd < 0
\]

(b) \[
\Delta U = -W > 0
\]

\(\Rightarrow U\) increases
2.1.2. Electric Potential and Electric Potential Difference:

- $U$ depends on $q$: $U \sim q$

$$V = \frac{U}{q} \quad \text{(unit: } J/C)$$

- $V$ is the potential energy per unit charge and it is characteristic only of the electric field, called the electric potential

- Electric potential difference $\Delta V$ between two points:

$$\Delta V = V_f - V_i = \frac{U_f}{q} - \frac{U_i}{q} = \frac{\Delta U}{q}$$

$$\Delta V = V_f - V_i = -\frac{W}{q}$$

- The potential difference between two points is the negative of the work done by the electrostatic force to move a unit charge from one point to other
• We set $U_i = 0$ at infinity:

$$V = -\frac{W_\infty}{q}$$

• We introduce a new unit for $V$:

1 volt = 1 joule per coulomb

• We adopt a new conventional unit for $E$:

$$1 \text{N/C} = \left(\frac{1\text{N}}{C}\right)\left(\frac{1\text{V.C}}{1\text{J}}\right)\left(\frac{1\text{J}}{1\text{N.m}}\right) = 1\text{V/m}$$

• We also define one electron volt that is the energy equal to the work required to move 1 e\(^-\) through a potential difference of exactly one volt:

$$1\text{eV} = 1.60 \times 10^{-19}\text{ J}$$
Work done by an applied force:

Suppose we move a particle of charge $q$ from point $i$ to $f$ in an electric field by applying a force to it:
The work-kinetic energy theorem gives:

$$\Delta K = K_f - K_i = W_{applied} + W$$

(or you can use: $\Delta K + \Delta U = W_{applied}$)

If $\Delta K = 0$ (the particle is stationary before and after the move):

$$W_{applied} = -W$$

$$\Delta U = U_f - U_i = W_{applied}$$

$$W_{applied} = q\Delta V$$
2.1.3. Equipotential Surfaces:

Concept: Adjacent points with the same electric potential form an **equipotential surface**.

No **net work** is done on a charged particle by an electric field between two points on the same equipotential surface.

Those surfaces are always perpendicular to electric field lines (i.e., to $\vec{E}$).
2.1.4. Finding the Potential from the Field:

**Problem:** Calculate the potential difference between two points \(i\) and \(f\)

\[
\begin{align*}
\text{If we choose } V_i &= 0: \\
V &= -\int_{i}^{f} \vec{E} \cdot d\vec{s} \\
V_f - V_i &= -\frac{W}{q_0} = -\int_{i}^{f} \vec{E} \cdot d\vec{s}
\end{align*}
\]
Checkpoint 3 (page 633):
The figure shows a set of parallel equipotential surfaces and 5 paths along which we shall move an electron.

(a) the direction of $E$
(b) for each path, the work we do positive, negative or zero?
(c) Rank the paths according to the work we do, greatest first.

(a) from the left to the right

$$V_f - V_i = -\frac{W}{q_0} = -\int_{i}^{f} \vec{E}.d\vec{s}$$

(b) negative: 4; positive: 1,2,3,5
(c) 3,1-2-5,4
2.2. Potential Difference in a Uniform Electric Field:

**Problem:** Find the potential difference between two points $V_f - V_i$:

$$V_f - V_i = \frac{W}{q_0} = -\int_{i}^{f} \vec{E}.d\vec{s}$$

The test charge $q_0$ moves along the path parallel to the field lines, so:

$$V_f - V_i = -Ed$$

- Electric field lines always point in the direction of decreasing electric potential
- Potential difference between two points does not depend on the path connecting them (electrostatic force is a conservative force)

**Check:** move $q_0$ following icf: 

$$V_f - V_i = -E(\cos 45^0) \frac{d}{\sin 45^0} = -Ed$$
2.3. Electric Potential and Potential Energy Due to Point Charges:

**Key equation:**

\[ V_f - V_i = -\int_{i}^{f} \vec{E} \cdot d\vec{s} \]

2.3.1. Potential Due to a Point Charge:

- Choose the zero potential at infinity
- Move \( q_0 \) along a field line extending radially from point \( P \) to infinity, so \( \theta = 0^\circ \)

\[
V_\infty - V_P = -\int_0^\infty E \, dr
\]

\[
V_P = kq \int_0^R \frac{1}{r^2} \, dr = k \frac{q}{R}
\]
In a general case:

\[ V = k \frac{q}{r} \]

A positively charged particle produces a positive electric potential, a negatively charged particle produces a negative electric potential.

Potentials \( V(r) \) at points in the xy plane due to a positive point charge at O.
2.3.2. Potential Due to A Group of Point Charges:

- Using the superposition principle:

\[ V = \sum_{i=1}^{n} V_i = k \sum_{i=1}^{n} \frac{q_i}{r_i} \]  

(an algebraic sum, not a vector sum)

Checkpoint 4: The figure shows three arrangements of two protons. Rank the arrangements according to the net electric potential produced at point P by the protons, greatest first.

Use of the formula above gives the same rank
2.3.3. Potential Due to an Electric Dipole:

**Problem:** Calculate $V$ at point $P$

$$V = \sum_{i=1}^{2} V_i = V_{(+)} + V_{(-)} = k \left( \frac{q}{r_{(+)}} + \frac{-q}{r_{(-)}} \right)$$

$$= kq \frac{r_{(-)} - r_{(+)}}{r_{(-)}r_{(+)}}$$

If $r >> d$:

$$r_{(-)} - r_{(+)} \approx d \cos \theta \text{ and } r_{(-)}r_{(+)} \approx r^2$$

$$V = k \frac{p \cos \theta}{r^2}; \quad p = qd$$
**Induced Dipole Moment:**

- **No external electric field:** in some molecules, the centers of the positive and negative charges coincide, thus no dipole moment is set up.

- **Presence of an external E:** the field distorts the electron orbits and hence separates the centers of positive and negative charge, the electrons tend to be shifted in a direction opposite the field. This shift sets up a dipole moment, the so-called **induced dipole moment**. The atom or molecule is called to be **polarized** by the field.
2.3.4. Calculating the Field from the Potential:

**Problem:** $q_0$ moves through a displacement $d\vec{s}$ from one equipotential surface to the adjacent surface:

\[ W = -q_0 dV = q_0 E (\cos \theta) ds \]

\[ E \cos \theta = -\frac{dV}{ds} \]

\( \frac{dV}{ds} \) : the gradient of the electric potential in the $d\vec{s}$ direction

$E_s = E \cos \theta$ is the component of $\vec{E}$ in the direction of $d\vec{s}$:

\[ E_s = -\frac{\partial V}{\partial s} \]

Partial Derivative
In Cartesian coordinates:

\[ E_x = -\frac{\partial V}{\partial x}; \quad E_y = -\frac{\partial V}{\partial y}; \quad E_z = -\frac{\partial V}{\partial z} \]

\[ \vec{E} = -\nabla V \]

\( \nabla \) or \( \vec{\nabla} \): gradient

\( \nabla \): Nabla symbol

\[ \nabla = \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k} \]
Two Dimensional Field and Potential

A uniformly charged rod

$V(x, y)$

A dipole

$V(x, y)$

2.3.5. Electric Potential Energy of a System of Point Charges:

\[ \Delta U = U_f - U_i = -W = W_{\text{applied}} \]

**Problem:** Calculate the electric potential energy of a system of charges due to the electric field produced by those same charges.

Consider a simple case: two point charges held a fixed distance \( r \).

**We define:** The electric potential energy of a system of fixed point charges is equal to the work that must be done by an external agent to assemble the system, bringing each charge in from an infinite distance.

The work by an applied force to bring \( q_2 \) in from infinity:

\[ W_{\text{applied}} = q_2 \Delta V = q_2 (V_f - V_\infty) \]

\[ W_{\text{applied}} = U_{\text{system}} = q_2 V = k \frac{q_1 q_2}{r} \]
If the system consists of three charges, we calculate $U$ for each pair of charges and sum the terms algebraically:

$$U = U_{12} + U_{13} + U_{23} = k\left(\frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}}\right)$$

### 2.4. Electric Potential Due to Continuous Charge Distributions

**Method:** We calculate the potential $dV$ at point $P$ due to a differential element $dq$, then integrate over the entire charge distribution

$$dV = k \frac{dq}{r}$$

$$V = \int dV = k \int \frac{dq}{r}$$
2.4.1. Line of Charge:

\[ dq = \lambda \, dx \]

\[ dV = k \frac{dq}{r} = k \frac{\lambda \, dx}{\left( x^2 + d^2 \right)^{1/2}} \]

\[ V = \int_0^L k \frac{\lambda \, dx}{\left( x^2 + d^2 \right)^{1/2}} \]

(see Appendix E, integral 17, page A-11)
2.4.2. Charged Disk:

\[ dq = \sigma(2\pi R')(dR') \]

\[ dV = k \frac{dq}{r} = k \frac{\sigma(2\pi R')(dR')}{\sqrt{z^2 + R'^2}} \]

\[ V = \frac{1}{4\pi \varepsilon_0} \int_0^R \sigma(2\pi R')(dR') \frac{1}{\sqrt{z^2 + R'^2}} \]

\[ V = \frac{\sigma}{2 \varepsilon_0} \left( \sqrt{z^2 + R^2} - z \right) \]
2.5. Electric Potential of a Charged Isolated Conductor:

Using Gauss’ law, we prove the following conclusion:

An excess charge placed on an isolated conductor will distribute itself on the surface of that conductor so that all points of the conductor (on the surface or inside) have the same potential.

\[ V_f - V_i = -\int \vec{E} ds \]

Electric field at the surface is perpendicular to the surface and it is zero inside the conductor, so:

\[ V_f = V_i \]
A large spark jumps to a car’s body and then exists by moving across the insulating left front tire, leaving the person inside unharmed.
Homework:

1, 6, 18, 24, 35, 43, 64 (page 648-653)
Overview

One goal of Physics is to provide the basic science for practical devices designed by engineers.

In this part, we study such a practical device called a capacitor in which electrical energy can be stored.
2.6. Capacitance. Capacitors in Parallel and in Series:

2.6.1. Capacitance:

Some concepts:

- A capacitor consists of two isolated conductors of any shape. These two conductors are called plates.

- A parallel-plate capacitor has two parallel conducting plates of area \( A \) separated by a distance \( d \) (symbol: \( \_|_\)|-\-\-\-\-\-\)

- When a capacitor is charged, its plates have equal but opposite charges of \(+q\) and \(-q\)
- All points on a plate have the same potential, \( V \) is the potential difference between the two plates.
- **For electrical devices:** symbol \( V \) (not \( \Delta V \)) often represents a potential difference.
- When a capacitor is charged:

\[
q = CV
\]

- \( C \): Capacitance of the capacitor.
- The capacitance describes how much charge an arrangement of conductors can hold for a given voltage applied and its value depends on the geometry (dielectric, plate area, distance) of the plates.
- Unit: 1 farad = 1 F = 1 Coulomb/Vol = 1 C/V
- \( 1 \mu F = 10^{-6} \) F; \( 1 \) pF = \( 10^{-12} \) F
Charging a capacitor:
- Place the capacitor in an electric circuit with a battery which is a device that maintains a certain potential difference between its terminals
- In schematic diagram (b) of a circuit: a battery B; a capacitor C; a switch S; interconnecting wires
- When S is closed: the electric field drives electrons from capacitor plate h to the positive terminal of B, plate h loosing e- becomes positively charged; the field drives electrons from negative terminal of B to plate l, so it gains electrons and becomes negatively charged
- \( V \) between the plates is equal to that of battery B

Checkpoint 1: Does the capacitance \( C \) of a capacitor increase, decrease, or remain the same (a) when the charge \( q \) on it is doubled and (b) when the potential difference \( V \) across it is tripled? (a) the same; (b) the same
2.6.2. Calculating the Capacitance:

**Goal:** Calculate the capacitance of a capacitor once we know its geometry.

\[ q = CV \]

**step 1:** Calculating the electric field

Use Gauss' law for a Gaussian surface as indicated:

\[ \varepsilon_0 \int \vec{E} d\vec{A} = q \Rightarrow q = \varepsilon_0 EA \]

**step 2:** Calculating the potential difference

\[ V = V_f - V_i = -\int_{i}^{f} \vec{E} d\vec{s} \]

If the integration path from the negative to the positive plates:

\[ V = \int_{-}^{+} Eds \]
(a) A Parallel-Plate Capacitor:

\[ q = \varepsilon_0 EA = CV \]

\[ V = \int_{d}^{0} E \, ds = E \int_{0}^{d} ds = Ed \]

\[ \Rightarrow C = \frac{\varepsilon_0 A}{d} \]

Using the formula above, we can use:

\[ \varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m} = 8.85 \text{ pF/m} \]

instead of

\[ \varepsilon_0 = 8.85 \times 10^{-12} \text{ C}^2/(\text{N.m}^2) \]
(b) A Cylindrical Capacitor:

- A cylindrical capacitor of length L includes two coaxial cylinders of radii a and b.
- We choose a Gaussian surface as indicated:

\[ q = \varepsilon_0 EA = \varepsilon_0 E(2\pi rL) \]

\[ E = \frac{q}{2\pi\varepsilon_0 rL} \]

\[ ds = -dr : \text{integrated radially inward} \]

\[ V = \int Eds = -\frac{q}{2\pi\varepsilon_0 L} \int_a^b \frac{dr}{r} = \frac{q}{2\pi\varepsilon_0 L} \ln\left(\frac{b}{a}\right) \]

\[ C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)} \]
(c) A Spherical Capacitor:

- A capacitor consists of two concentric spherical shells of radii a and b

\[ q = \varepsilon_0 E A = \varepsilon_0 E (4\pi r^2) \]

\[ E = \frac{1}{4\pi \varepsilon_0} \frac{q}{r^2} \]

\[ V = \int_{+}^{-} Eds = -\frac{q}{4\pi \varepsilon_0} \int_{b}^{a} \frac{dr}{r^2} = \frac{q}{4\pi \varepsilon_0} \left( \frac{1}{a} - \frac{1}{b} \right) \]

\[ C = 4\pi \varepsilon_0 \frac{ab}{b - a} \]
(d) An Isolated Sphere:

A capacitor consists only a single isolated spherical conductor of radius \( R \), assuming that the missing plate is a conducting sphere of infinite radius.

\[
C = 4\pi \varepsilon_0 \frac{ab}{b-a} = 4\pi \varepsilon_0 \frac{a}{1 - \frac{a}{b}}
\]

\[
C = 4\pi \varepsilon_0 R
\]

Checkpoint 2: For capacitors charged by the same battery, does the charge stored by the capacitor increase, decrease, or remain the same in each of the following situations? (a) The plate separation of a parallel-plate capacitor is increased. (b) The radius of the inner cylinder of a cylindrical capacitor is increased. (c) The radius of the outer spherical shell of a spherical capacitor is increased.

(a) decreases; (b) increases; (c) decreases
Capacitance Summary

- **Unit:** 1 farad = 1 F = 1 Coulomb/Vol = 1 C/V
- 1 µF = 10^{-6} F; 1 pF = 10^{-12} F

- **Parallel-Plate Capacitor:**
  \[ C = \frac{\varepsilon_0 A}{d} \]

- **Cylindrical Capacitor:**
  \[ C = 2\pi\varepsilon_0 \frac{L}{\ln(b/a)} \]

- **Spherical Capacitor:**
  \[ C = 4\pi\varepsilon_0 \frac{ab}{b-a} \]

- **Isolated Sphere:**
  \[ C = 4\pi\varepsilon_0 R \]
  \[ \varepsilon_0 = 8.85 \times 10^{-12} \text{ F/m} = 8.85 \text{ pF/m} \]
2.6.3. Capacitors in Parallel and in Series:

If there is a combination of capacitors in a circuit, we can replace that with an equivalent capacitor, which has the same capacitance as the actual combination of capacitors.

(a) Capacitors in Parallel:

A potential difference $V$ is applied across each capacitor. The total charge $q$ stored on the capacitors is the sum of the charges stored on all capacitors:

$$ q = q_1 + q_2 + q_3 = (C_1 + C_2 + C_3)V $$

$$ C_{eq} = \frac{q}{V} = C_1 + C_2 + C_3 $$

If we have $n$ capacitors in parallel:

$$ C_{eq} = \sum_{i=1}^{n} C_i $$
(b) Capacitors in Series:

- All capacitors have identical charges $q$
- The sum of the potential differences across the capacitors is equal to the applied potential difference $V$

$$V = V_1 + V_2 + V_3 = q \left( \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \right)$$

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}$$

If we have $n$ capacitors in series:

$$\frac{1}{C_{eq}} = \sum_{i=1}^{n} \frac{1}{C_i}$$
Checkpoint: A battery of potential \( V \) stores charge \( q \) on a combination of two identical capacitors. What are the potential difference across and the charge on either capacitor if the capacitors are (a) in parallel and (b) in series?

\[
\begin{align*}
V, \ q/2 \\
\end{align*}
\]

\[
\begin{align*}
V/2, \ q \\
\end{align*}
\]
2.7. Energy Stored in a Charged Capacitor:

- When we charge a capacitor, work must be done by an external agent and the work is stored in the form of electric potential energy $U$ in the electric field between the plates.
- At any instant, the capacitor has $q'$ and $V' = q'/C$, if an extra charge $dq'$ is then transferred:

$$dW = V' dq' = \frac{q'}{C} dq' \quad \iff \quad W = \int dW = \frac{1}{C} \int_0^q q' dq' = \frac{q^2}{2C}$$

The potential energy of the capacitor:

$$U = \frac{q^2}{2C} \quad \quad U = \frac{1}{2} CV^2$$
The medical defibrillator:
A medical device is used to stop the fibrillation of heart attack victims based on the ability of storing potential energy of a capacitor.

**Example:** a 70 µF capacitor in a defibrillator is charged to 5000 V giving an amount of energy $U$:

$$U = \frac{1}{2} CV^2 = 875 \text{ (J)}$$

About 200 J is sent through the victim in a pulse of 2 ms, so the power is:

$$P = \frac{U}{t} = 100 \text{ (kW)}$$
Energy density:

Consider a parallel-plate capacitor, the uniform energy density \( u \), the potential energy per unit volume, between the plates:

\[
u = \frac{U}{Ad} = \frac{CV^2}{2Ad}
\]

\[
C = \frac{\varepsilon_0 A}{d} \quad \Rightarrow \quad u = \frac{1}{2} \varepsilon_0 \left( \frac{V}{d} \right)^2
\]

The electric field is uniform between the plates (neglecting fringing):

\[
E = -\frac{\Delta V}{\Delta s} = \frac{V}{d}
\]

\[
\Rightarrow \quad u = \frac{1}{2} \varepsilon_0 E^2
\]
2.8. Capacitor with a Dielectric:

**Some concepts:**

Dielectric: an insulating material such as mineral oil or plastic

- If we fill the space between the plates of a capacitor with a dielectric, its capacitance increases by a factor $\kappa$ (kappa), which is called the dielectric constant (for a vacuum, $\kappa = 1$)
- A dielectric has a maximum potential difference $V_{\text{max}}$ that can be applied between the plates, called breakdown potential.
- Every dielectric has a characteristic dielectric strength $E_{\text{max}}$ corresponding to $V_{\text{max}}$

\[
C' = \kappa C_{\text{air}}
\]

- $C'$: capacitance with a dielectric
- $C_{\text{air}}$: air between the plates
If $V$ is maintained (e.g., by a battery), more charge flows to the capacitor with a dielectric.

If $q$ is maintained, voltage drops in the capacitor with a dielectric.
2.8.1. Dielectrics: An Atomic View
(a) What happens when we put a dielectric in an external electric field?
+ For polar dielectrics: the molecules already have electric dipole moments (e.g., water)
+ For nonpolar dielectrics: in a field, molecules acquire dipole moments by induction
So the molecules tend to line up with the field, Their alignment produces an electric field in the opposite direction to the external field and thus reduces the applied field.

In a region completely filled by a dielectric material, all electrostatic equations containing ε₀ are to be corrected by replacing ε₀ with κε₀.
The presence of a dielectric will therefore reduce the difference potential for a given charge \((q = \text{constant})\) or will increase the charge on the plates.

### 2.8.2. Dielectrics and Gauss' Law:

**Goal:** Generalize Gauss's law if dielectrics are present

We consider a parallel-plate capacitor, without a dielectric:

\[
\varepsilon_0 \int \vec{E} d\vec{A} = \varepsilon_0 E_0 A = q
\]
\[ E_0 = \frac{q}{\varepsilon_0 A} \]

- If a dielectric is present:

\[ \varepsilon_0 \int \vec{E}d\vec{A} = \varepsilon_0 EA = q - q' \]

\[ E = \frac{q - q'}{\varepsilon_0 A} \]

- The effect of the dielectric is to reduce the applied field \( E_0 \):

\[ E = \frac{E_0}{\kappa} = \frac{q}{\kappa \varepsilon_0 A} \Rightarrow q - q' = \frac{q}{\kappa} \]

\[ \varepsilon_0 \int \kappa \vec{E}d\vec{A} = q \]

or

\[ \int \vec{D}d\vec{A} = q; \vec{D} = \varepsilon_0 \kappa \vec{E} : \text{electric displacement} \]
Checkpoint: Two identical parallel-plate capacitors are connected in series to a battery as shown below. If a dielectric is inserted in the lower capacitor, which of the following increase for that capacitor?
(a) Capacitance of the capacitor 
(b) Charge on the capacitor 
(c) Voltage across the capacitor 
(d) Energy stored in the capacitor

\[
C = \kappa C_0 \quad q = CV \quad U = \frac{1}{2} CV^2
\]

(a) the capacitance \( C_2 \) increases by \( \kappa \), \( C_2 = \kappa C \)
(b) more charge flows to \( C_2 \) as \( C_{	ext{equivalent}} \) increases; \( q_2 \) increases \( q_1 = q_2 \) also increases, \( q' = \frac{2\kappa q}{\kappa + 1} \)
(c) \( q_1 \) increases, so \( V_1 \) increases; \( V_1 + V_2 = V = \text{constant}, V_2 \downarrow \)
(d) before: \( U = \frac{q^2}{2C} \) after: \( U = \frac{q^2}{2C} \times \frac{4\kappa}{(\kappa + 1)^2} < \frac{q^2}{2C} \) so, \( U \downarrow \)
Homework:

2, 6, 11, 16, 26, 31, 33, 42, 48, 51 (page 676-680)