Chapter 4:

Linear Momentum and Collisions

4.1. The Center of Mass, Newton's Second Law for a System of Particles

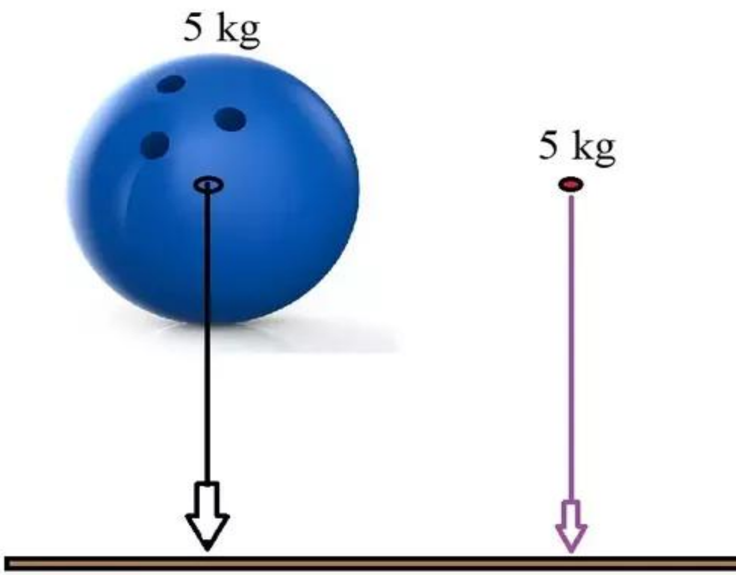
4.2. Linear Momentum and Its Conservation

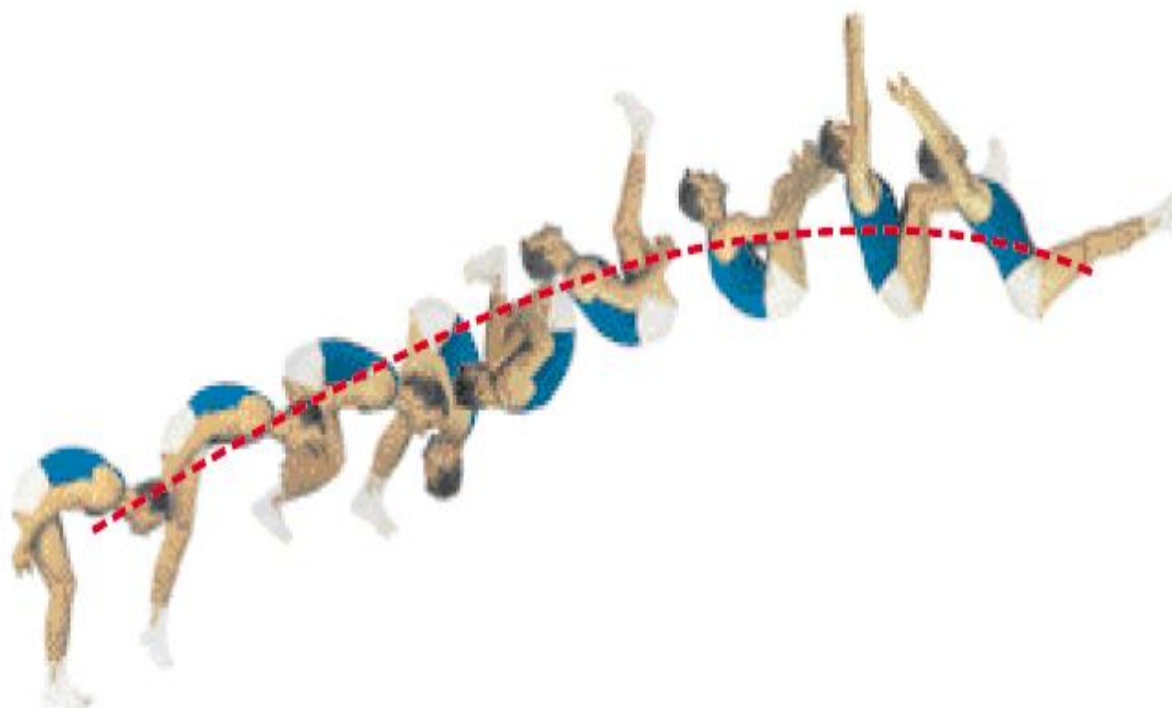
4.3. Collision and Impulse

4.4. Momentum and Kinetic Energy in Collisions

4.1. The Center of Mass. Newton's Second Law for a System of Particles

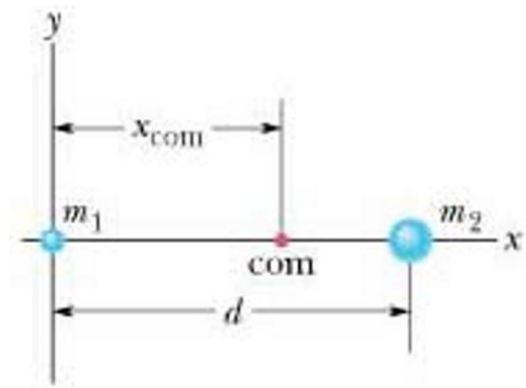
4.1.1. The center of mass (COM)





4.1.1. The center of mass

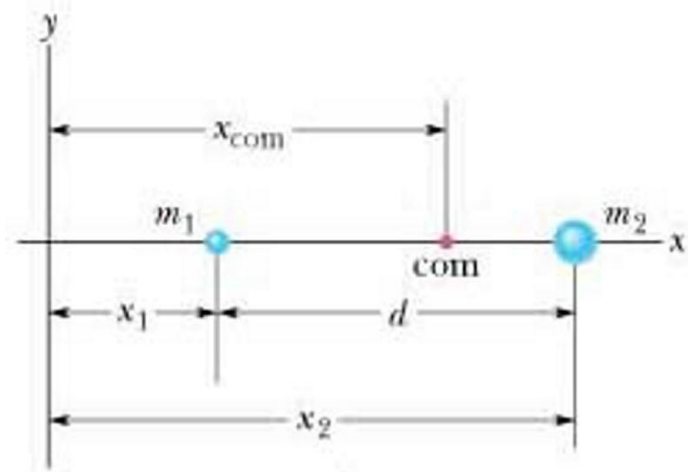
a. Systems of Particles



(a)

•Consider a system of 2 particles of masses m_1 and m_2 separated by distance d:

$$x_{com} = \frac{m_2}{m_1 + m_2} d$$



(b)

•If m_1 at x_1 and m_2 at x_2:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2} = \frac{m_1 x_1 + m_2 x_2}{M}$$

where M is the total mass of the system

•If the system has n particles that are strung out along the x axis:

$$x_{com} = \frac{m_1 x_1 + m_2 x_2 + \ldots + m_n x_n}{M} = \frac{1}{M} \sum_{i=1}^{n} m_i x_i$$

- If the n particles are distributed in three dimensions:

$$x_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i x_i, \quad y_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i y_i, \quad z_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i z_i$$

- If the position of particle i is given by a vector:

$$\vec{r}_i = x_i\hat{i} + y_i\hat{j} + z_i\hat{k}$$

$$\vec{r}_{com} = x_{com}\hat{i} + y_{xom}\hat{j} + z_{com}\hat{k}$$

- The center of mass of the system is determined by:

$$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i\vec{r}_i$$

b. Solid Bodies

$$x_{com} = \frac{1}{M}\int x\,dm, \quad y_{com} = \frac{1}{M}\int y\,dm, \quad z_{com} = \frac{1}{M}\int z\,dm$$

where M is the mass of the object

- **For uniform objects, their density are:**

$$\rho = \frac{dm}{dV} = \frac{M}{V}$$

$$\Rightarrow dm = \left(\frac{M}{V}\right)dV$$

$$x_{com} = \frac{1}{V}\int x\,dV, \quad y_{com} = \frac{1}{V}\int y\,dV, \quad z_{com} = \frac{1}{V}\int z\,dV$$

Sample Problem (p. 204)

Determine the center of mass of the plate

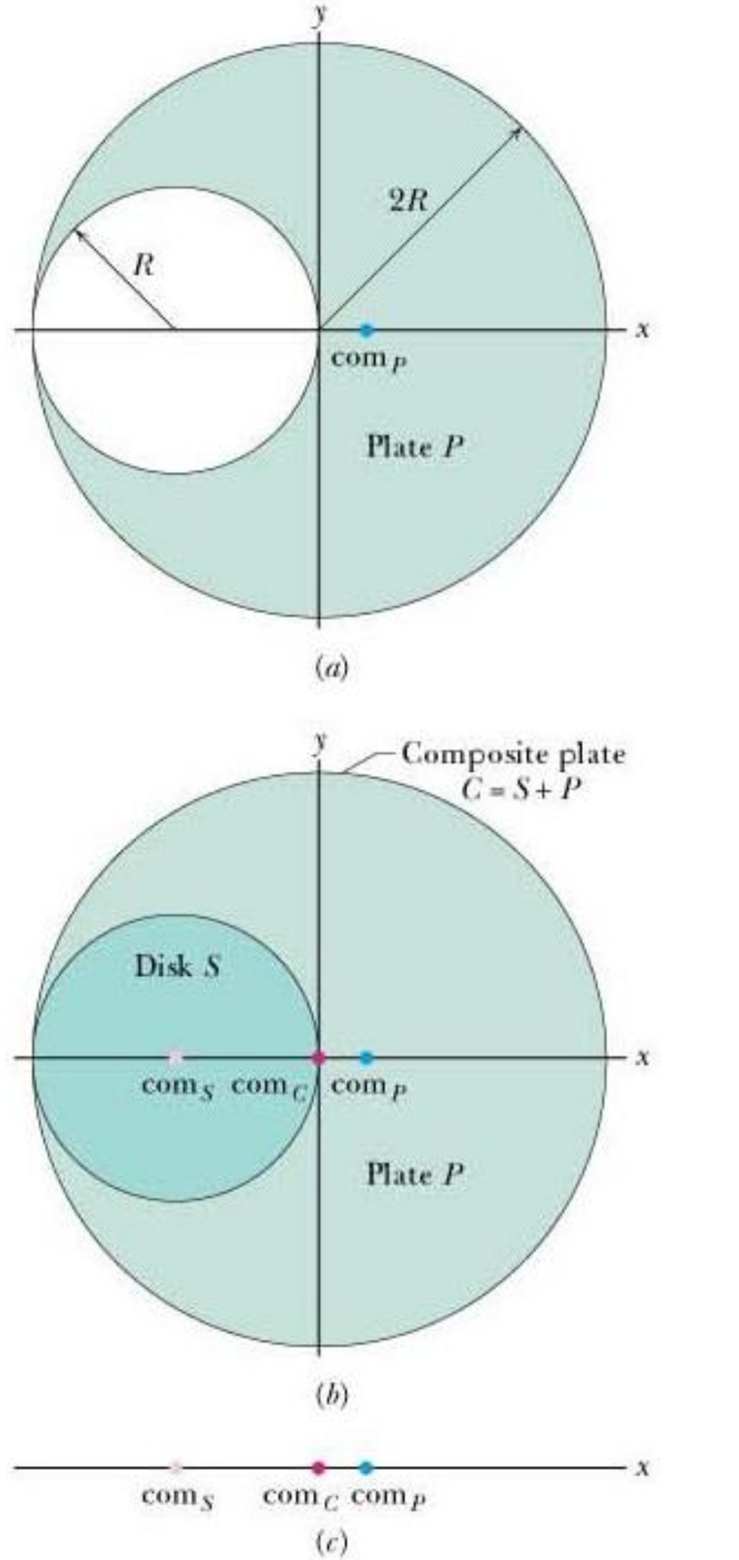
$$x_{S+P} = \frac{m_S x_S + m_P x_P}{m_S + m_P} = 0$$

$$x_P = -x_S \frac{m_S}{m_P}$$

$$\frac{m_S}{m_P} = \frac{\rho_S}{\rho_P} \times \frac{thickness_S}{thickness_P} \times \frac{area_S}{area_P} = \frac{\pi R^2}{\pi (2R)^2 - \pi R^2} = \frac{1}{3}$$

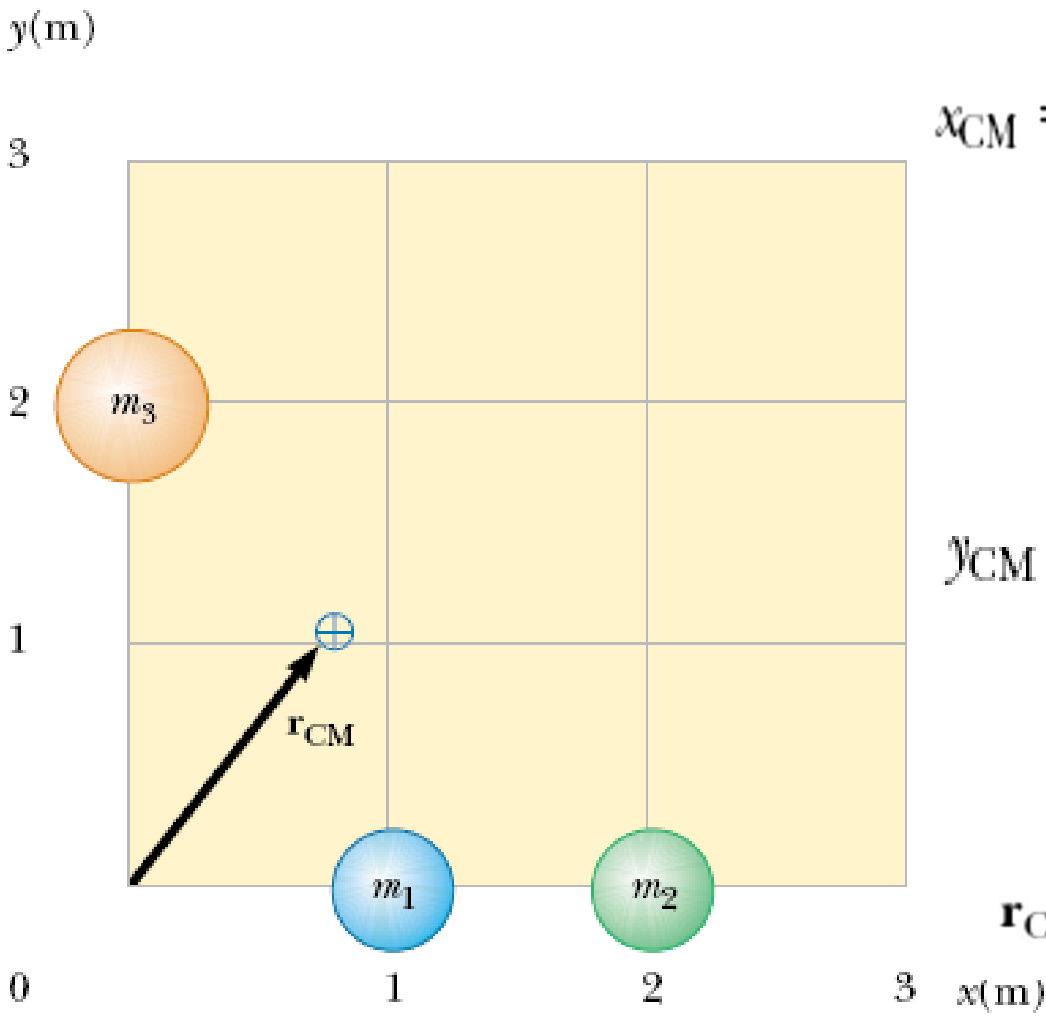
$$x_S = -R$$

$$\Rightarrow x_P = \frac{1}{3} R$$



Sample Problem

A system consists of three particles located as shown in the figure. Find the center of mass of the system.



$$x_{CM} = \frac{\sum_i m_i x_i}{M} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3}$$

$$= \frac{3.0 \text{ kg} \cdot \text{m}}{4.0 \text{ kg}} = 0.75 \text{ m}$$

$$y_{CM} = \frac{\sum_i m_i y_i}{M} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3}$$

$$= \frac{4.0 \text{ kg} \cdot \text{m}}{4.0 \text{ kg}} = 1.0 \text{ m}$$

$$\mathbf{r}_{CM} = x_{CM}\mathbf{i} + y_{CM}\mathbf{j} = 0.75\mathbf{i} \text{ m} + 1.0\,\mathbf{j} \text{ m}$$

4.1.2. Newton's Second Law for a System of Particles

$$\vec{F}_{net} = M\vec{a}_{com} \qquad (1)$$

$\vec{F}_{net}$: the net force for all external forces

$\vec{a}_{com}$: the acceleration of the center of mass of the system.

M : the total mass of the system.

$$F_{net,x} = Ma_{com,x} \qquad F_{net,y} = Ma_{com,y} \qquad F_{net,z} = Ma_{com,z}$$

4.2. Linear Momentum and Its Conservation

a. Linear Momentum

The linear momentum of a particle is a vector quantity $\vec{p}$ defined as:

$$\vec{p} = m\vec{v}$$

(Unit: kg m/s)

where m and $\vec{v}$ are the mass and the velocity of the particle, respectively.

Newton's second law is expressed in terms of momentum:

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}$$

where $\vec{F}_{net}$ is the net external force on the particle.

•For a system of particles:

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \ldots + \vec{p}_n = m_1\vec{v}_1 + m_2\vec{v}_2 + \ldots + m_n\vec{v}_n$$

$$\vec{P} = M\vec{v}_{com} \quad \Rightarrow \frac{d\vec{P}}{dt} = M\frac{d\vec{v}_{com}}{dt} = M\vec{a}_{com}$$

→The linear momentum of a system of particles is equal to the product of the total mass M of the system and the velocity of the center of mass.

b. Conservation of Linear Momentum:

If the net external force acting on a system of particles is zero,

$$\vec{F}_{net} = 0 \qquad \vec{P} = \text{constant}$$

$$\text{If } F_{net,X} = 0 \text{ (X = x, y, or z) :} \qquad P_X = \text{constant}$$

Question: Why do we need momentum?

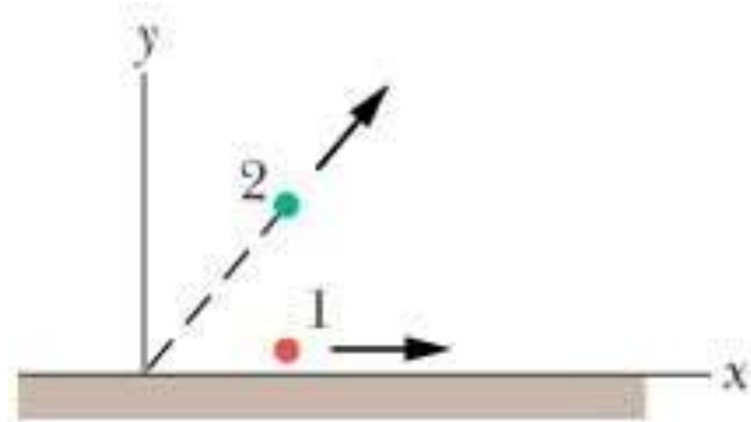
Because momentum provides us a tool for studying collision of 2 or more objects.

14. Two particles are launched from the origin of the coordinates system at time t=0. m_1=5.0 g is shot directly along the x axis with a constant speed of 10 m/s. m_2=3.0 g is shot with a velocity of magnitude 20.0 m/s, at an upward angle such that it always stays directly above particle 1 during its flight.

(a) What is the maximum height H_{max} reached by the COM of the two particle system?

(b) In unit-vector notation, what are the (b) velocity and (c) acceleration of the com when the com reaches H_{max}?

$$v_{2,y}^2 - v_{2,y0}^2 = -2gy$$



(a) At the maximum height:

$$-v_{2,y0}^2 = -2gy_{\max}$$

Particle 2 always stays directly above P.1:

$$v_{2,x} = v_{1,x}$$

$$\Rightarrow v_{2,y0} = \sqrt{v_2^2 - v_{2,x}^2} = \sqrt{v_2^2 - v_{1,x}^2} = 17.3\,(\text{m/s})$$

$$y_{\max} = 15.3\,(\text{m}) \Rightarrow H_{\max} = \frac{m_2 y_{\max}}{m_1 + m_2} = 5.74\,(\text{m})$$

(b) $$\vec{v}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{v}_i = \frac{m_1\vec{v}_1 + m_2\vec{v}_2}{m_1 + m_2}$$

At the maximum height, $v_{2,y}$=0:

$$v_{com,y} = 0;\ v_{com,x} = \frac{m_1 v_{1,x} + m_2 v_{2,x}}{m_1 + m_2} = v_{1,x}$$

$$\vec{v}_{com} = (10\ \text{m/s})\,\hat{\text{i}}$$

(c) $$M\vec{a}_{com} = \sum_{i=1}^{n} m_i \vec{a}_i$$

$$\vec{a}_{com} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2}$$

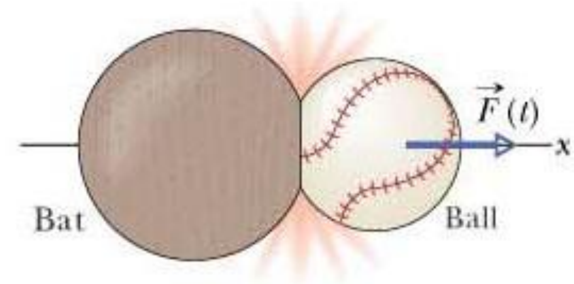
$$a_{com} = \frac{m_2 g}{m_1 + m_2} = 3.68\,(\text{m/s}^2)$$

$\vec{a}_{com}$ is downward, hence :

$$\vec{a}_{com} = (-3.68\ \text{m/s}^2)\ \hat{\text{j}}$$

4.3. Collision and Impulse

- Consider a collision between a bat and a ball:



The change in the ball's momentum is:

$$d\vec{p} = \vec{F}(t)dt$$

from a time t_i to a time t_f:

$$\int_{t_i}^{t_f} d\vec{p} = \int_{t_i}^{t_f} \vec{F}(t)dt$$

- The impulse of the collision is defined by:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt$$

(Unit: kg m/s)

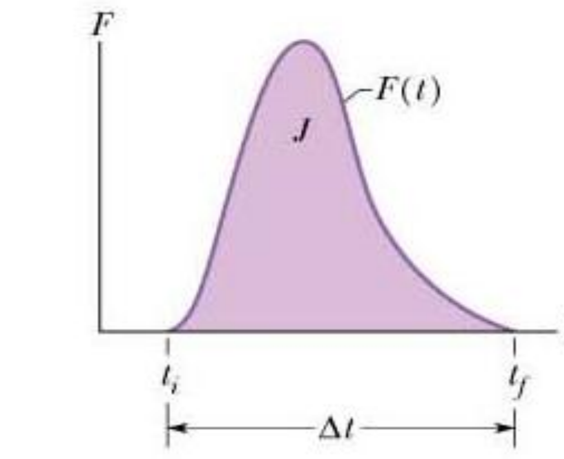
$$\Delta\vec{p} = \vec{J}$$

the impulse of the object

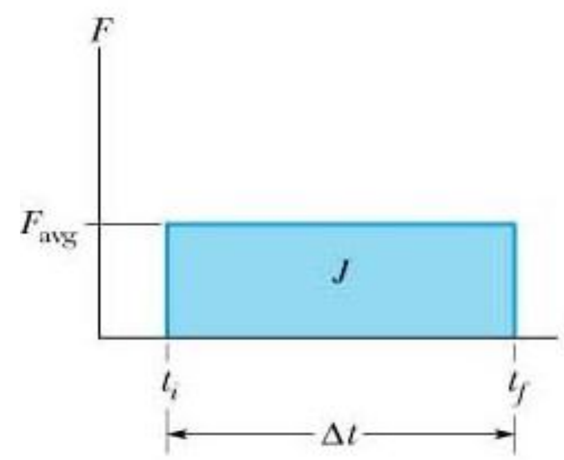
the change in the object's momentum

If F(t) function is unknown:

$$J = F_{avg}\Delta t$$



(a)



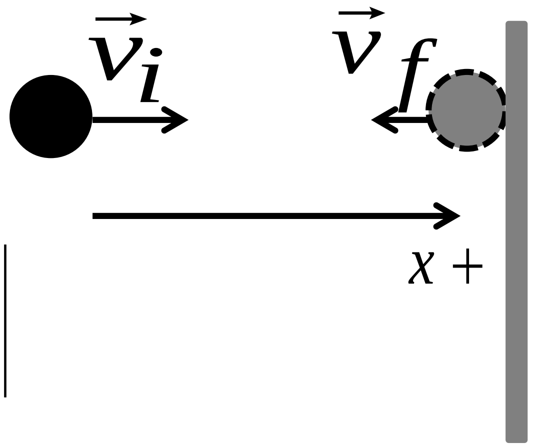
(b)

Examples:

1. A 0.70 kg ball is moving horizontally with a speed of 5.0 m/s when it strikes a vertical wall. The ball rebounds with a speed of 2.0 m/s. What is the magnitude of the change in linear momentum of the ball?

$$\vec{p} = m\vec{v}; \quad \Delta\vec{p} = m\Delta\vec{v}$$

Since the ball is moving horizontally, therefore, this is one dimensional motion:



$$\Delta p_x = m\Delta v_x$$

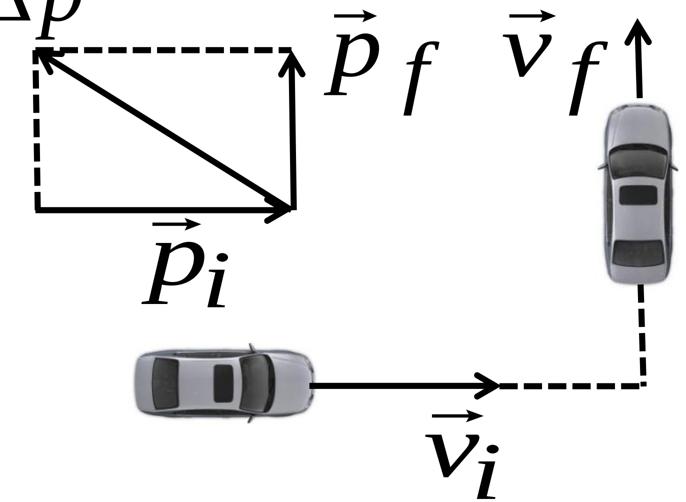
$$|\Delta p_x| = m|\Delta v_x| = m|(v_f - v_i)|$$

$$v_f = -2 \text{ m/s}; \; v_i = 5 \text{ m/s} : |\Delta p_x| = |0.7 \times (\text{-}2 - 5)| = 4.9 \text{ (kg m/s)}$$

$$\Delta\vec{p} = m\Delta\vec{v} = (-4.9 \text{ kg.m/s})\,\hat{i}$$

2. A 1500-kg car travelling at a speed of 5.0 m/s makes a 90^0 turn in a time of 3.0 s and emerges from this turn with a speed of 3.0 m/s:
(a) What is the magnitude of the impulse that acts on the car during this turn? Draw the impulse vector.
(b) What is the magnitude of the average force on the car during this turn? (Final exam, June 2014)

$$\vec{J} = \Delta\vec{p}$$



(a)

$$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i$$

$$J = \sqrt{p_f^2 + p_i^2} \approx 8746 \text{ (kg m/s)}$$

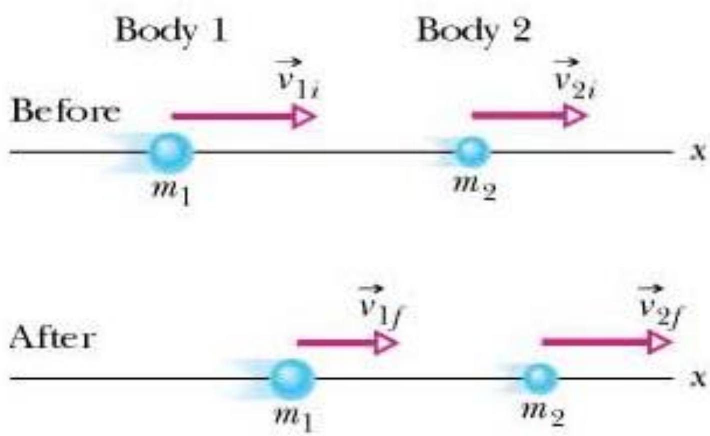
(b)

$$F_{\text{avg}} = \frac{J}{\Delta t} = \frac{8746}{3} \approx 2915 \text{ (N)}$$

4.4. Momentum and Kinetic Energy in Collisions

Three types of collisions: We consider a system of 2 bodies

1. Inelastic collision:



$$\text{total momentum} : \vec{P} = \text{constant}$$

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

$$KE \neq \text{constant}$$

Some energy (KE) is transferred to other forms, e.g. heat, sound.

$$\vec{v}_{com} = \frac{\vec{P}}{m_1 + m_2} = \frac{\vec{p}_{1i} + \vec{p}_{2i}}{m_1 + m_2} = \frac{\vec{p}_{1f} + \vec{p}_{2f}}{m_1 + m_2} = \text{constant}$$

2. Elastic collision: $\vec{p}$ and KE are conserved.

$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f}$$

$$K_{1i} + K_{2i} = K_{1f} + K_{2f}$$

- **In one dimension:** $m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f}$

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2$$

Special cases:

- $v_{2i} = 0$:

$$v_{1f} = \frac{m_1 - m_2}{m_1 + m_2} v_{1i}$$

$$v_{2f} = \frac{2m_1}{m_1 + m_2} v_{1i}$$

$+ m_1 = m_2$: $\quad v_{1f} = 0; \quad v_{2f} = v_{1i}$

$+ m_2 >> m_1$: $\quad v_{1f} \approx -v_{1i}; \quad v_{2f} \approx \left(\frac{2m_1}{m_2}\right) v_{1i}$

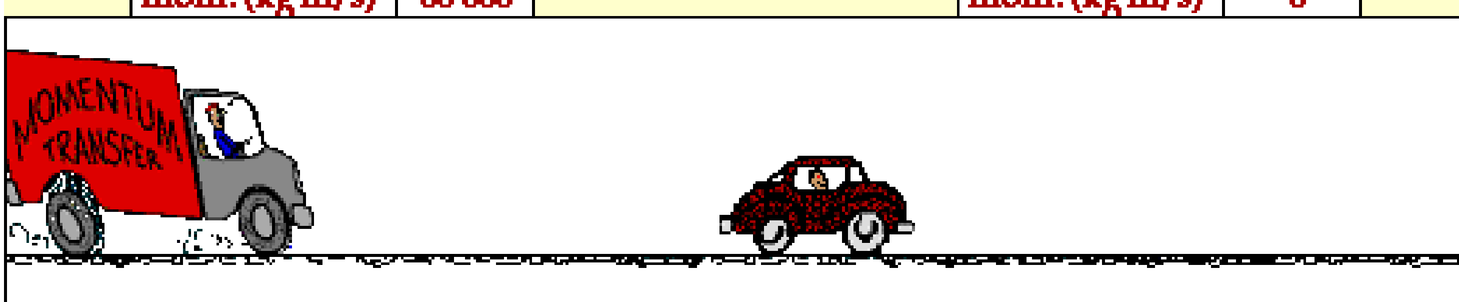
$+ m_1 >> m_2$: $\quad v_{1f} \approx v_{1i}; \quad v_{2f} \approx 2v_{1i}$

Truck

mass (kg)	3000
vel. (m/s)	20.0
mom. (kg m/s)	60 000

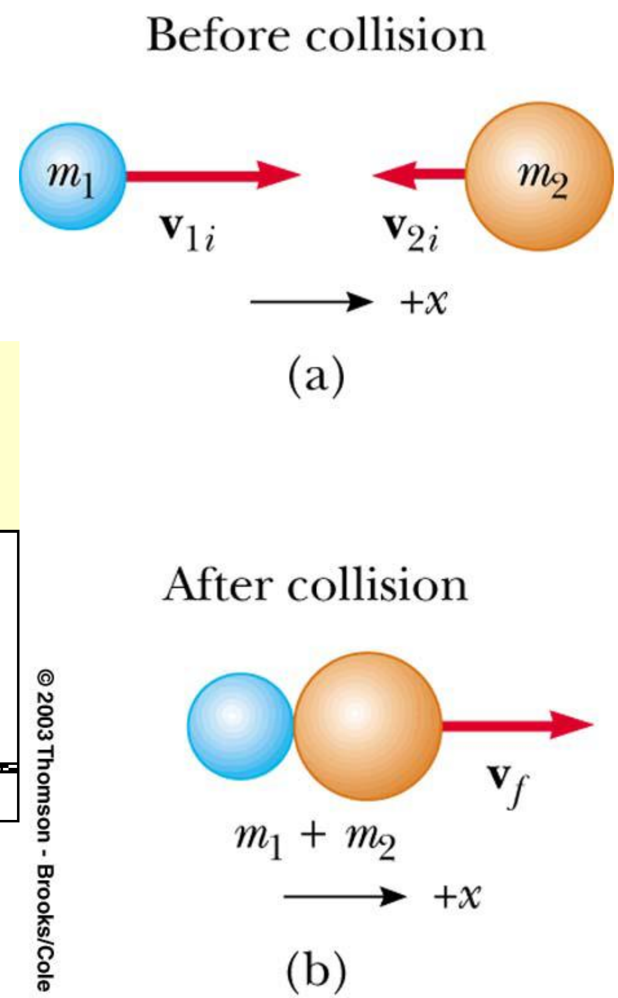
Car

mass (kg)	1000
vel. (m/s)	0.0
mom. (kg m/s)	0



3. Perfectly inelastic collision: two bodies stick together after collision:

$\vec{p}$ conserved but not KE.



3.1. In one dimension:

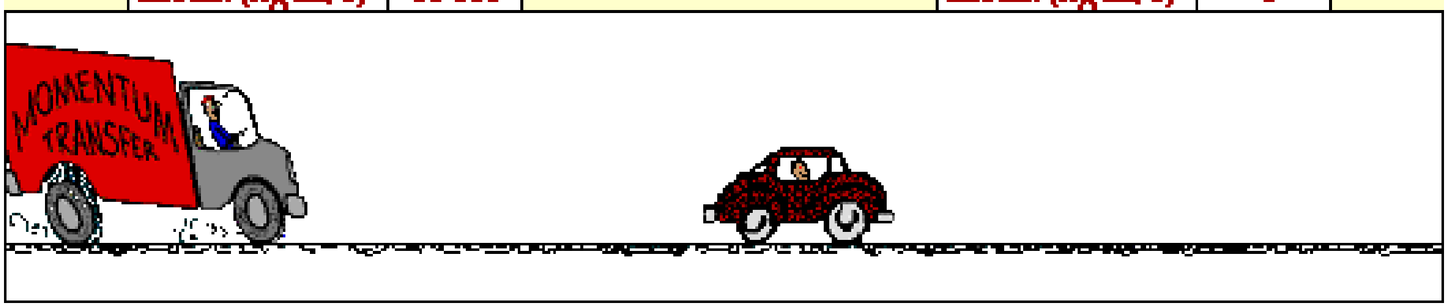
- $v_{1f} = v_{2f} = v_f$: $m_1 v_{1i} + m_2 v_{2i} = (m_1 + m_2) v_f$

Truck

mass (kg)	3000
vel. (m/s)	20.0
mom. (kg m/s)	60 000

Car

mass (kg)	1000
vel. (m/s)	0.0
mom. (kg m/s)	0



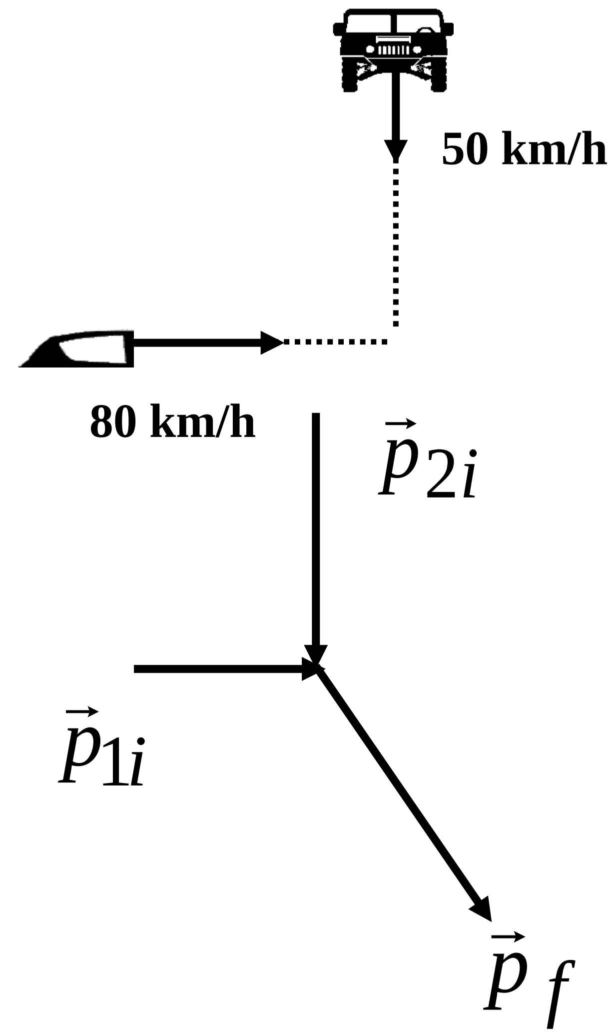
3.2. In two dimensions:

$$m_1 \vec{v}_{1i} + m_2 \vec{v}_{2i} = (m_1 + m_2) \vec{v}_f$$

Case 3

Example: (Perfectly inelastic collision)

A 1000-kg car travelling east at 80.0 km/h collides with a 3000 kg car traveling south at 50.0 km/h. The two cars stick together after the collision. What is the speed of the cars after the collision? (Final exam, June 2014)



$$\vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_f$$

$$m_1\vec{v}_{1i} + m_2\vec{v}_{2i} = (m_1 + m_2)\vec{v}_f$$

$$p_f = \sqrt{p_{1i}^2 + p_{2i}^2} = \sqrt{(1000 \times 80.0)^2 + (3000 \times 50.0)^2}$$

$$p_f = 170000 \ \text{(kg km/h)}$$

$$v_f = \frac{p_f}{(m_1 + m_2)} = 42.5 \ \text{(km/h) or } 11.8 \ \text{m/s}$$

Homework:

2, 5, 13, 14, 22, 25, 38, 49, 56, 67, 60, 64, 74 (pages 230-237)

Review:

1. Center of mass: $$\vec{r}_{com} = \frac{1}{M}\sum_{i=1}^{n} m_i \vec{r}_i$$

2. Linear Momentum $$\vec{p} = m\vec{v}$$

Unit: kg m/s

$$\vec{F}_{net} = \frac{d\vec{P}}{dt}$$

3. Impulse: $$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)\,dt$$

$$\Delta\vec{p} = \vec{J}$$

$$J = F_{avg}\Delta t$$

4. Conservation of Linear Momentum: $$\vec{P} = \text{constant}$$

(a closed, isolated system)

5. Momentum and Kinetic Energy in Collisions

- **Inelastic Collisions:** $$\begin{cases} \vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \\ KE \neq \text{constant} \end{cases}$$

- **Elastic Collisions:** $$\begin{cases} \vec{p}_{1i} + \vec{p}_{2i} = \vec{p}_{1f} + \vec{p}_{2f} \\ K_{1i} + K_{2i} = K_{1f} + K_{2f} \end{cases}$$